# ANOMALIES

- Anomalies are consequence of quantum mechanics and symmetries.
- An anomaly is certain symmetry, modified by quantum effects, in a subtle and precise way.

⇒ Consequences

(i) Calculable physical effects ;

(ii) Nontrivial constraints on nonAbelian gauge theory (the gauge group, the matter content, etc. )

(ii) Nontrivial consistency conditions in composite models for quarks and leptons

N.B. quarks and electrons, muons, tau
<u>are</u> they <u>really</u> <u>elementary</u>?

## $U_A(1)$ and chiral anomalies.

$(D_\mu = \partial_\mu + ieA_\mu)$

QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\gamma^\mu D_\mu - m)\psi$$

$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$, $\sigma^\mu = (-1, \sigma^i)$, $\bar\sigma^\mu = (1, \sigma^i)$

$\delta\mathcal{L} = 0$ under $\psi \to e^{i\alpha}\psi$ : $U_V(1)$

- charge conservation
- local $U_V(1)$ invariance $\alpha = \alpha(x_\mu)$
- $\psi \to e^{i\alpha\gamma_5}\psi$ : $U_A(1)$ (global)

$$\Rightarrow J^\mu = \bar\psi\gamma^\mu\psi \; ; \quad J_5^\mu = \bar\psi\gamma^\mu\gamma^5\psi$$

$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \mathbb{1} & \\ & -\mathbb{1} \end{pmatrix}$

$$\partial_\mu J^\mu = 0 \; ; \quad \partial_\mu J_5^\mu = im\bar\psi\gamma^5\psi \xrightarrow[m\to 0]{} 0$$

$$\mathcal{L}_{QCD} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \bar\psi_i(\gamma^\mu D_\mu - m_i)\psi_i \; ; \; i = 1, 2 \ldots N_f$$

$$J^\mu = \sum_i \bar\psi_i\gamma^\mu\psi_i \; ; \quad J_5^\mu = \sum_i \bar\psi_i\gamma^\mu\gamma_5\psi_i$$

$$J^{a\mu} = \bar\psi^i (T^a)_i^j \gamma^\mu \psi_j \; ; \quad J_5^{a\mu} = \bar\psi^i (T^a)_i^j \gamma^\mu\gamma_5\psi_j$$

$m_i = 0$

$$\partial_\mu J^{a\mu} = \partial_\mu J_5^{a\mu} = 0 \; ; \quad \partial_\mu J^\mu = \partial_\mu J_5^\mu = 0$$

The theory is classically inv. under

$SU_V(N_f) \times SU_A(N_f) \times U_V(1) \times U_A(1)$, or

$SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times U_A(1)$,

$$\psi_L = \frac{(1+\gamma_5)}{2}\psi \; ; \quad \psi_R = \frac{1-\gamma_5}{2}\psi .$$

# Chirality

- Fundamental entity of Nature ~ chiral fermion, ( $e_L, e_R, \nu_L$ ; $q_L, q_R$ .. )
  (Feynman, Gell-Mann)
- Dirac fermion ~ $\psi_L + \psi_R$ ( $L^{\uparrow} \sim SU(2)\times SU(2)$ (2, 1) (1, 2) )
- Parity violation in (electro) weak interactions
- Chiral symmetry: quarks $u_L, u_R, d_L, d_R$

  $G_F = SU_L(2) \times SU_R(2) \times U_L(1) \times U_R(1)$
- Broken spontaneously ( $\langle \bar{u}_R u_L \rangle = \langle \bar{d}_R d_L \rangle \neq 0$ )

  to $G_{Manifest} = \underset{isospin}{SU(2)} \times U_B(1)$

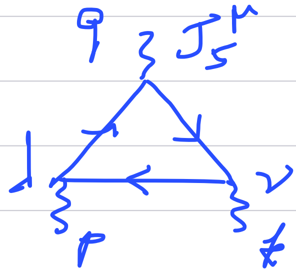
# Axial anomaly.

Fukuda, Miyamoto, '49
Steinberger '49
Schwinger '51
Adler, Jackiw

$$\psi \to e^{i\gamma_5 \alpha}\psi$$

$$\partial_\mu J_5^{\ \mu} = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (*) \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \quad (\text{dual of } F_{\mu\nu})$$

Triangle anomaly:




$$G^{\mu;\lambda\nu} \equiv \langle 0|T\{J_5^\mu(x) J^\lambda(y) J^\nu(0)\}|0\rangle =$$

$$\sim \int d^4\ell \frac{1}{\ell^3} \sim \text{linearly divergent.}$$

$\ell_\mu \to \ell_\mu + r_\mu$ change the value of the integral.

Require $\partial_\lambda G^{\mu;\lambda\nu} = 0$; $\partial_\nu G^{\mu;\lambda\nu} = 0$;
then one finds $\partial_\mu G^{\mu;\lambda\nu} \neq 0$ such that $(*)$ holds

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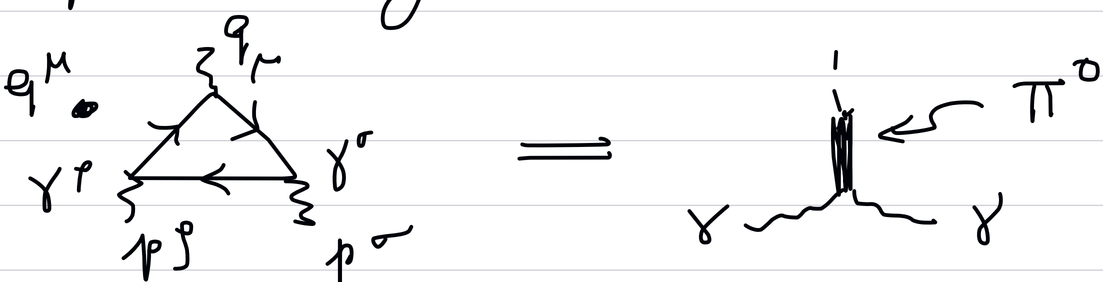
Triangle anomaly and $\pi^0 \to \gamma\gamma$.




$$\therefore \Gamma(\pi^0 \to 2\gamma) = \frac{\alpha^2}{64\pi^2}\frac{m_\pi^3}{F_\pi^2} N_c^2\left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right]^2 \simeq 1.10^{-5}\,\text{MeV}$$

cfr $\Gamma^{exp}(\pi^0 \to 2\gamma) \simeq \frac{\hbar}{\tau} \simeq \frac{1}{1.2}\cdot 10^{-5}$ MeV

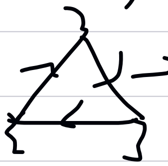
**N.B.** $N_c^2 \sim 10$ essential for $\sim 20\%$ agreement.

# $\pi^0$ decay rate

$$A^{\mu\nu} \equiv \langle \overset{\text{photons}}{\epsilon^\mu(p), \epsilon^\nu(k)} | \partial_\mu J_5^\mu(q) | 0 \rangle$$

$$= \langle \epsilon^\mu(q), \epsilon^\nu(k) | \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu} | 0 \rangle = \frac{e^2}{16\pi^2} C \cdot 2\epsilon_{\mu\nu\rho\sigma} p^\rho k^\sigma$$

where $C = 2N_c \left[ \underbrace{\left(\frac{2}{3}\right)^2}_{u} + \underbrace{\left(-\frac{1}{3}\right)^2}_{d} \right]$




On the other hand,

$$A^{\mu\nu} \simeq \underbrace{\langle \epsilon^\mu(p), \epsilon^\nu(k) | \pi \rangle}_{A_{\pi\to\gamma\gamma}} \frac{1}{q^2} \langle \pi(q) | \partial_\mu J_5^\mu | 0 \rangle$$

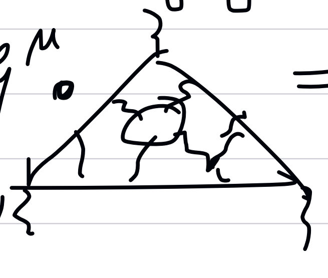
$\langle \pi(q) | J_5^\mu | 0 \rangle =: F_\pi q^\mu$ (pion decay constant)

$$A_{\pi\to\gamma\gamma} = \frac{N_c e^2}{4\pi^2 F_\pi} \left[ \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 \right] \epsilon_{\mu\nu\rho\sigma} p^\rho k^\sigma$$

$\Rightarrow \Gamma_{\pi\to\gamma\gamma} \simeq 1 \cdot 10^{-5}$ MeV $\left( F_\pi \simeq 180 \text{ MeV} \leftarrow \pi \to \mu\nu \right)$

## Significance of such a one-loop calculation?

- One-loop anomaly (*) is exact.
- Sutherland-Veltman theorem:

$$\sum_{\text{graphs}} q^\mu \cdot$$  $$= 0$$

# Origin of anomaly.

- Locality (and causality) of fundamental Interactions = fact of Nature. (String theory?)
- Divergencies. ⇒ Necessity of renormalization (paradigm of contemporary particle physics.)

1. Regularize each operator (precise procedure for removing UV divergencies) ; → finite # of
2. Fit the values of certain reference quantities to experimental data (e.g. $e$)
3. Remove the regularization (UV cutoff, $\epsilon$, ...)
4. If the procedure works to all orders, the theory is renormalizable.
5. Well-defined predictions order by order

Axial current operator (point-splitting method).

$$J_5^\mu = \bar{\psi}(x)\gamma^5\gamma^\mu\psi(x) = \lim_{\epsilon\to 0} \bar{\psi}\left(x+\frac{\epsilon}{2}\right)\gamma^5\gamma^\mu \left( e^{\,ie\int_{x-\epsilon/2}^{x+\epsilon/2} dx^\mu A_\mu} \right) \psi\left(x-\frac{\epsilon}{2}\right)$$

1. $\frac{1}{\epsilon}$ divergences when inserted in a graph;
2. $O(\epsilon)$ contribution from the string bit;
3. Compute the finite $\epsilon \cdot \frac{1}{\epsilon}$ piece;
4. Send $\epsilon \to 0$.

$$\Longrightarrow \quad \partial_\mu J_5^\mu = \frac{e^2}{16\pi^2} F_{\mu\nu}\tilde{F}^{\mu\nu}.$$

Other regularizations (give the same results)

1) Momentum space subtractions (BPHZ)
2) Dimensional regularization
3) Pauli-Villars
4) Functional-integral (Fujikawa)

Phys. Rev. D21 (1980), 2848

Use $\quad i\gamma^\mu \partial_\mu \psi = m\psi - e\gamma_\mu A^\mu \psi \quad$ (eq of motion)

$$\Rightarrow \partial_\mu J_5^\mu = \underset{(= im\,\bar\psi\gamma_5\psi)}{J_5} - ie\, J_5^\mu(x,\epsilon)\cdot\Big[A_\mu(x+\tfrac{\epsilon}{2}) - A_\mu(x-\tfrac{\epsilon}{2})$$

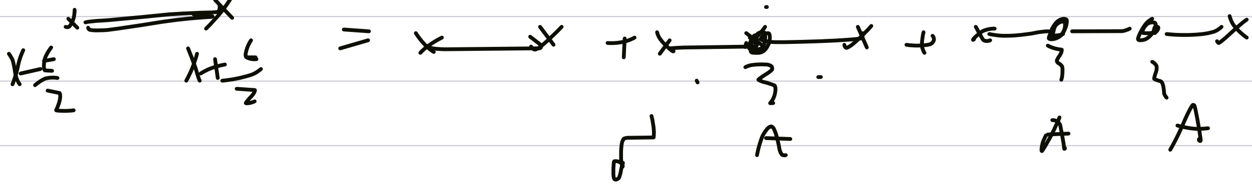
$$- \partial_\mu \int_{x-\epsilon/2}^{x+\epsilon/2} A_\nu(y)\,dy^\nu \Big]$$

$$= \underbrace{J_5} - ie\, J_5^\mu\, \epsilon^\alpha\left[\partial_\alpha A_\mu - \partial_\mu A_\alpha + O(\epsilon)\right]$$

$$\langle \partial_\mu J_5^\mu \rangle = \langle J_5 \rangle - ie\,\epsilon^\alpha \cdot F_{\alpha\mu}(x).$$

$$\int D\psi D\bar\psi\; \bar\psi(x+\tfrac{\epsilon}{2})\, i\gamma^\mu\gamma_5\, \psi(x-\tfrac{\epsilon}{2}).\; e^{i\int d^4x\; i\bar\psi\gamma^\mu(\partial_\mu - ieA_\mu)\psi +}$$

$$= J_5 - ie\,\epsilon^\alpha F_{\alpha\mu}\cdot \mathrm{Tr}\,\gamma^\mu\gamma_5 \langle \psi(x-\tfrac{\epsilon}{2})\,\bar\psi(x+\tfrac{\epsilon}{2})\rangle$$




$$\int d^4z\; \langle \psi(x+\tfrac{\epsilon}{2})\,\bar\psi(z)\rangle\, \gamma^\rho A_\rho(z)\, \langle \psi(z)\,\bar\psi(x-\tfrac{\epsilon}{2})\rangle$$

$$\sim \mathrm{Tr}(\gamma_5 \gamma^\mu \gamma^\rho \gamma^\lambda \gamma^\sigma)$$

$$\int d^4 z \int d^4 p \, e^{ip(x+\frac{\epsilon}{2}-z)} \frac{\cancel{p}_\rho}{p^2} A_\lambda(z) \int dq \, \frac{e^{iq\cdot(z-x+\frac{\epsilon}{2})}}{q^2} q_\lambda$$

$$\to A(x) + (x-z)^\kappa \partial_\kappa A_\lambda + \cdots$$

$$\int d^4 z \, \frac{1}{(z-x-\frac{\epsilon}{2})^3} A_\lambda(z) \frac{1}{(z-x+\frac{\epsilon}{2})^3}$$

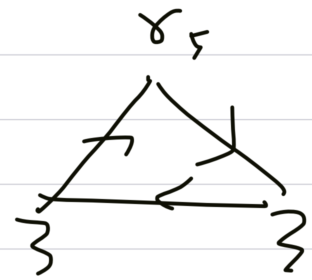
$$\sim \frac{1}{\epsilon} \cdot F_{\kappa\lambda}$$

Pauli-Villars regulators.

$$\partial_\mu J_5{}^\mu = 2mi\,\bar\psi\gamma^5\psi - 2Mi\,\underbrace{\bar\psi_M \gamma^5 \psi_M}.$$

$$\downarrow$$

$$M \cdot \frac{1}{M} \sim \text{finite}.$$



# Fujikawa's derivation of $U_A(1)$ anomalies.

$$W[A] = e^{-\Gamma(A)} = \int D\psi D\bar\psi \, e^{-\int dx \bar\psi i\gamma\cdot D^A \psi}$$

$x^0 \to -ix^4$ ; $A_0 \to iA_4$

$$D\psi D\bar\psi = \prod_n da_n d\bar b_n$$

$\not{D} = \gamma^\alpha D_\alpha = \gamma^\alpha(\partial_\alpha + A_\alpha)$

$\downarrow$

$SU(N_f)$

$$i\gamma\cdot D^{(A)}\psi_n = \lambda_n \psi_n \quad \text{and}$$

$$\psi = \sum a_n \psi_n \ ; \quad \bar\psi = \sum_n \bar b_n \psi_n^\dagger \ :$$

change of variables

$$\psi \to e^{i\alpha(x)\gamma_5}\psi \ ; \quad \bar\psi \to \bar\psi e^{i\alpha(x)\gamma_5} \ ;$$

$$\psi = \psi' + \underline{i\alpha(x)}\gamma_5\psi' \ ; \quad \bar\psi = \bar\psi' + i\alpha(x)\bar\psi'\gamma_5$$

$$\psi = \cancel{\sum a_n \psi_n} = \sum_n a_n e^{-i\alpha}\gamma_5\psi_n' = \sum a_n' \psi_n'$$

$$\therefore \quad da_m' = \sum_n \left[\int dx \, \psi_m^*(1+\alpha(x)\gamma_5)\psi_n\right] da_n = \sum_n C_{mn} da_n$$

$$\prod_m da_m' = \det C \prod da_n \ ; \quad \prod_n da_n = (\det C)^{-1} \prod_m da_m'$$

analog.ly

$$\prod_n d\bar b_n = \prod_n d\bar b_n' \, (\det C)^{-1}$$

$$\therefore \quad J = (\det C)^{-2}$$

$$\det C = e^{\log\det C} = e^{\mathrm{Tr}\log(1+A)} \simeq e^{\mathrm{Tr}\,A}$$

$$\therefore \quad J = \exp -2i\int dx \, \alpha(x) \sum_n \psi_n^*(x)\gamma_5\psi_n(x)$$

Naively, $\sum_n \psi_m^*(x)\gamma_5\psi_n(x) = \mathrm{Tr}\,\gamma_5 = 0$, (Really, $0\cdot\infty = ?$)

# Regularization

$$\sum_n \psi_n^* \gamma_5 \psi_n(x) \equiv \lim_{M\to\infty} \sum_n \psi_n^* \gamma_5\, e^{-\lambda_n^2/M^2} \psi_n(x)$$

$$= \lim_M \sum_n \psi_n^* \gamma_5\, e^{-(\gamma\cdot D)^2/M^2} \psi_n = \lim_M \mathrm{Tr}\left[\gamma_5\, e^{-(\gamma\cdot D)^2/M^2}\right]$$

Compute now

$$\mathrm{Tr}(\ldots) \to \int \frac{d^4p}{(2\pi)^4}\, e^{ipx}\, \mathrm{tr}(\ldots)\, e^{-ip\cdot x}$$

Now $(\gamma\cdot D)^2 = D_\mu^2 + \sigma_{\mu\nu}F^{\mu\nu}$, $\sigma_{\mu\nu} \equiv \frac{\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu}{4}$

$$D_\mu = \partial_\mu - igA_\mu = i(P_\mu - gA_\mu) \;;\; P_\mu = -i\partial_\mu$$

$$D_\mu^2 = -(P_\mu - gA_\mu)^2 \equiv -P_\mu^2 - \Delta$$

$$\therefore\ \mathrm{Tr}\,\gamma_5\, e^{(\gamma\cdot D)^2/M^2} = \int \frac{d^4p}{(2\pi)^4}\, e^{ip\cdot x}\, \mathrm{tr}\,\gamma_5\, e^{(p^2+\Delta-\sigma_{\mu\nu}F^{\mu\nu})/M^2} e^{-ip\cdot x}$$

$$= \lim \frac{1}{2}\, \mathrm{tr}\,\gamma_5\, \sigma_{\mu\nu}\sigma_{\rho\sigma} \frac{F^{\mu\nu}F^{\rho\sigma}}{M^4} \cdot \int \frac{d^4p}{(2\pi)^4}\, e^{-p^2/M^2}$$

$$= \lim \frac{1}{2}\, \mathrm{Tr}\, \frac{F_{\mu\nu}F_{\rho\sigma}}{M^4}\, \frac{4}{16}\, \mathrm{tr}(\gamma_5\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma)\, \frac{1}{(2\pi)^4} \frac{\pi^2}{\Gamma(2)} \int_0^\infty dp^2\, p^2\, e^{-p^2/M^2}$$

$$\Rightarrow \mathrm{Tr}\, \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}F_{\rho\sigma} = \underbrace{\frac{1}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}}$$

$$\Big( iA_\mu = gA_\mu^a T^a \;;\; [T^a, T^b] = i f^{abc} T^c$$

$$\mathrm{tr}\, T^a T^b = \tfrac{1}{2}\delta_{ab} \;;\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \Big)$$

Chiral anomalies, Non Abelian anomalies.

$$e^{-\Gamma(A)} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, e^{-\int dx\, \bar{\psi}_L i\gamma_\mu D^\mu \psi_L}$$

$$\psi_L \equiv \frac{1+\gamma_5}{2}\psi$$

gauge transf.:

$$A_\mu \to A_\mu - D_\mu v, \quad D_\mu v = \partial_\mu v + ig[A_\mu, v]$$

$$\Gamma(A) \to \Gamma(A) + \int d^4x\, v^a D_\mu \frac{\delta\Gamma(A)}{\delta A^a_\mu}.$$

If the theory is inv.

$$\frac{\delta\Gamma}{\delta v^a} = 0 \quad \therefore \quad D_\mu \langle J^a_\mu \rangle = 0, \quad J^a_\mu \equiv \frac{\delta\Gamma(A)}{\delta A^a_\mu}.$$

Actually, the variation is nonvanishing: (W. Bardeen)

$$\delta_{v_L}\Gamma(A) = \frac{1}{24\pi^2}\int d^4x\, \mathrm{Tr}\, v(x)\, \epsilon^{\mu\nu\alpha\beta}\partial_\nu\left(A_\mu \partial_\alpha A_\beta + \frac{1}{2}A_\mu A_\alpha A_\beta\right)$$

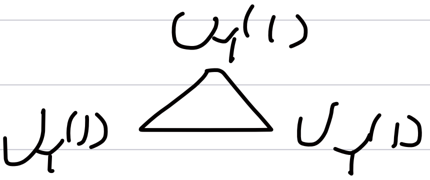
$$= \frac{1}{24\pi^2}\int d^4x \int \mathrm{Tr}\, v(x)\, d\left(AdA + \frac{1}{2}A^3\right) \propto \mathrm{Tr}\, T^a\{T^b, T^c\} \equiv d^{abc}(r)$$

$$v \equiv v^a T^a$$

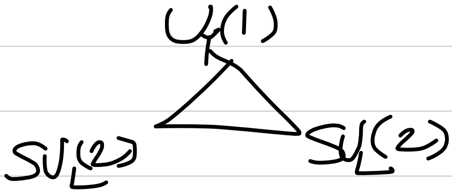
- $A_\mu$ = electroweak gauge fields
  → Wess-Zumino-Witten effective action for $\Gamma_{eff}(U, A)$
  ↳ Nambu-Goldstone bosons of $SU(3)\times SU(3)/SU(3)$
- $A_\mu = SU(3)\times SU_L(2)\times U(1)$ gauge fields
  → Consistency conditions (Gauge ~~anomalies~~)

# Cancellation of anomalies in the standard model

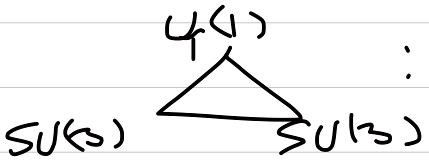
The standard model is <u>chiral</u> $\Rightarrow$
Algebraic cancellation of gauge anomalies.




$$\left(\tfrac{1}{3}\right)^3\cdot 2\cdot 3-\left(\tfrac{4}{3}\right)^3\cdot 3-\left(-\tfrac{2}{3}\right)^3\cdot 3+(-)^3\cdot 2-(-2)^3$$
$$=\frac{2}{9}-\frac{64}{9}+\frac{8}{9}-2+8=-6+6=0 \quad !$$




$$\left(\tfrac{1}{3}\right)\cdot 3\cdot 1-1=1-1=0.$$




: $\quad \frac{1}{3}\cdot 2-\left[\frac{4}{3}-\frac{2}{3}\right]=0$

- Quarks and leptons are both needed.
  (in complete "families")

Wess-Zumino consistency cond. ; Witten's quantization

The anomaly is a variation of the effective action

so $(\delta_{v_1} \cdot \delta_{v_2} - \delta_{v_2} \cdot \delta_{v_1}) \Gamma(A) = \delta_{[v_1, v_2]} \Gamma(A)$

- $U \sim$ NG bosons of $SU(3) \times SU(3) / SU(3) \sim SU(3)$

$\pi_5(SU(3)) \sim \mathbb{Z}$ $\therefore n \in \mathbb{Z}$ $(n = N_c = 3)$

- Local action on the boundary of $S^4 \sim \mathbb{R}^4$

$$\Gamma_{eff} = n \cdot \int d\Sigma_{i_1 i_2 \cdots i_5} \, \omega_{i_1 \cdots i_5} \equiv n \Gamma(U)$$

$$\omega_{i_1 \cdots i_5} = -\frac{i}{240\pi^2} \mathrm{Tr}\left( (U^{-1}\partial_{i_1} U)(U^{-1}\partial_{i_2} U) \cdots (U^{-1}\partial_{i_5} U) \right)$$

$\rightarrow \Gamma_{eff}(U, A)$ (e.g., $A(\pi_0 \rightarrow \gamma\gamma)$, $N_c = 3$)

(Gauging).

# 't Hooft's anomaly matching condition

- $|p\rangle = |uud\rangle$ ; $|n\rangle = |udd\rangle$
- quarks and leptons : truely elementary? Or composite of more fundamental constituents?

e.g. $\quad q \sim PPP \leftarrow$ preons.

If $P \sim \underline{r}$ of $G_F$ then

$$q \subset \underline{r} \otimes \underline{r} \otimes \underline{r}$$

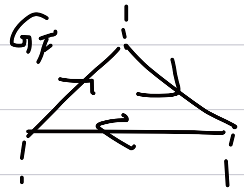
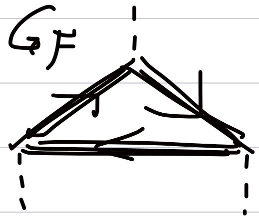
- Fermions are in chiral representations of $G_F$
- Suppose $G_F$ are weakly gauged. Introduce "leptons"

| | $G_C \otimes G_F$ | $G_F$ anomaly | |
|---|---|---|---|
| preons | $\underline{N_C} \otimes \underline{R}$ | $A_P$ | $A_P + A_\ell = 0$ |
| "leptons" | $\underline{1} \otimes R_\ell$ | $A_\ell$ | |

- At low energies

| | | | |
|---|---|---|---|
| Composites (quarks) | $\underline{1} \otimes R_q$ | $A_q$ | $A_q + A_\ell = 0$ |
| "leptons" | $1 \otimes R_\ell$ | $A_\ell$ | |

$\therefore \quad A_q = A_P$

$\sum_{\text{preons}}$  $=$ $\sum_{\text{quarks}}$ 

- $$\sum_{\text{preons}} d^{abc}(\underline{r}) = \sum_{\text{quarks}} d^{abc}(\underline{r_q})$$

- Dynamical results (#, types of light composite fermions) must obey the algebraic conditions

- Deep connections $<$ Dynamics / Symmetries

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Anomaly matching and $\pi_0 \to \gamma\gamma$.

(i) Both consequences of axial (chiral) anomalies

(ii) The same "anomaly" content in the high-energy and low-energy theories.

(iii) The anomaly $\Rightarrow$ singularity at $q^2 = 0$ of the three-point functions

(iv) Singularity $(1/q^2)$ either due to

$\left\{ \begin{array}{l} \text{a) massless NG bosons } (G_F \text{ spontaneously broken}); \\ \text{b) due to pairs of massless fermions (unbroken } G_F); \\ \text{c) both (partially due to NG bosons, \& to fermions)} \end{array} \right.$

# Anomaly and dynamics of QCD

$U_A(1)$ anomaly in the presence of nonAbelian gauge interactions

$$\partial_\mu J_5^\mu = \frac{g^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu}, \quad J_5^\mu = \bar{q}\gamma_5\gamma^\mu q$$

↳ gauge bosons. (gluons)

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$$

$$\pi_3(G_c) = \pi_3(SU(2)) = \mathbb{Z} \quad : \quad \text{instantons (YM)}$$

$$\int d^4x\, \partial_\mu J_5^\mu = \int d^3x\, J_5^0(t=\infty) - \int d^3x\, J_5^0(t=-\infty)$$

$$= \Delta Q_5 = \int d^4x \frac{g^2}{32\pi^2} G^a_{\mu\nu}\tilde{G}^{a\mu\nu} = +1 \quad !$$

"U(1)" problem $\quad SU_L(2)\times SU_R(2)\times U_V(1)\times U_A(1) \to SU_V(2)\times U_V(1)$

$\Rightarrow$ 4 NG bosons to be expected.

$\exists$ Only three NG candidate $\pi^\pm, \pi^0$,

$m_\pi \sim 140$ MeV ; $m_\eta \sim 500$ MeV

Instanton soln in SU(2) gauge theory.

YM action

$$S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{a\mu\nu}, \quad a=1,2,3.$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$$

$$(f^{abc} = \epsilon^{abc} \text{ for } SU(2)).$$

adding fermions

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi} i\gamma^\mu(\partial_\mu - igA_\mu)\psi \quad (*)$$

$$A_\mu \equiv T^a A^a_\mu \qquad \left(T^a = \frac{\tau^a}{2},\ SU(2)\right)$$

$$F_{\mu\nu} \equiv F^a_{\mu\nu} T^a = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

The action w. $(*)$ is inv under the gauge tr.,

$$\begin{cases} \psi \to U\psi \\ A_\mu \to U\cdot\left(A_\mu + \frac{i}{g}\partial_\mu\right)U^\dagger \end{cases} \qquad U = \exp\left(i\alpha^a(x)T^a\right)$$

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# Instantons in $SU(2)$ (YM) theories.

$$e^{-W} = \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \, e^{-S}$$

$$S = S_{gauge} + S_{matter}$$

$$S_{gauge} = \frac{1}{2}\int d^4x \, Tr(G_{\mu\nu}G_{\mu\nu}) ; \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

$$S_{matter} = \sum_{quarks} i\bar{\psi}\gamma^\mu(\partial_\mu - igA_\mu)\psi$$

- Finite action fields

$$A_\mu \to i U^\dagger \partial_\mu U \quad \text{f} \quad |x| \to \infty$$

$$Tr\, G_{\mu\nu}G_{\mu\nu} \to 0$$

Classified by : $\quad |x| = R \to \infty$

$$A : S^3 \to SU(2) \sim S^3, \quad \pi_3(SU(2)) = \mathbb{Z}$$

- "Winding" # = Pontryagin number

$$q = \frac{g^2}{32\pi^2}\int d^4x \, G^a_{\mu\nu}\tilde{G}^a_{\mu\nu} \in \mathbb{Z}, \quad \left(\tilde{G}^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^a_{\rho\sigma}\right)$$

It's a surface term

$$= \int_{R^4} \partial_\mu K^\mu d^4x ; \quad K^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\sigma} Tr\left(G_{\alpha\beta}A_\gamma - \frac{2}{3}A_\alpha A_\beta A_\gamma\right)$$

Instanton

$$S = \int \frac{1}{2} \mathrm{Tr}\, G_{\mu\nu} G^{\mu\nu} = \frac{1}{4} \int \mathrm{Tr} \left[ \left( G_{\mu\nu} - \tilde{G}_{\mu\nu} \right)^2 + 2 G_{\mu\nu} \tilde{G}^{\mu\nu} \right]$$

$$\Longrightarrow \frac{1}{4} \int d^4x \, \mathrm{Tr} \left( G_{\mu\nu} - \tilde{G}_{\mu\nu} \right)^2 + \frac{1}{2} \int d^4x \, \mathrm{Tr}\, G_{\mu\nu} \tilde{G}_{\mu\nu}$$

$$\geq \frac{8\pi^2}{g^2} \cdot n \qquad n = 0, 1, 2, \cdots$$

Search for solns of (Belavin Polyakov Shwartz Tyupkin)

$$G_{\mu\nu} = \pm \tilde{G}_{\mu\nu} \quad (\text{self-dual}) \; ; \quad S^{\mathrm{inst}} = \frac{8\pi^2}{g^2}$$

$$A_\mu^{\mathrm{inst}} = \frac{\tau^a}{2} A_\mu^a = \frac{2i}{g} \frac{\tau_{\mu\nu} (x - x_0)^\nu}{(x - x_0)^2 + \rho^2} \; ; \quad \left( \tau_\mu = (1, -i\sigma_i) \right)$$